JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 21, No. 3, May–June 1998

Control Allocation and Zero Dynamics

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Closed-loop stability for dynamic inversion controllers depends on the stability of the zero dynamics. The zero dynamics, however, depend on a generally nonlinear control allocation function that optimally distributes redundant controls. Therefore, closed-loop stability depends on the control allocation function. A sufficient condition is provided for globally asymptotically stable zero dynamics with a class of admissible nonlinear control allocation functions. It is shown that many common control allocation functions belong to the class of functions that are covered by the aforementioned zero dynamics stability condition. Aircraft flight control examples are given to demonstrate the utility of the results.

I. Introduction

SYSTEMS with more inputs than outputs are said to have redundant controls and are referred to as redundant systems, whereas systems with equal numbers of inputs and outputs are referred to as square systems. Redundant systems differ from square systems in that the control redundancy may be used to satisfy additional control system requirements such as minimal control effort, minimal computational complexity, control group prioritization, or preferred controls. There are many names in the literature that refer to the manner in which control redundancy is addressed. These include but are not limited to control management, control distribution, control selection, and the term that will be used in this paper, control allocation. Regardless of the name, control allocation is most important in these days of existing system upgrades as opposed to new system development. In the area of fighter aircraft flight control, upgrades will no doubt include additional control effectors.[1] There has been research in the area of novel control effectors including thrust vectoring,[2–5] pneumatic flow manipulators,[6] and movable nose strakes.[7] Even more recently, there is research involving other innovative control effectors such as all-moving wing tips.[8] Control redundancy, and thus control allocation, is inherent with the addition of these new control effectors. Constrained or limited control allocation involves allocation of commands to actuators that have limited ranges of motion. Constrained control allocation has been around for years, and Buffington and Enns[9] provide a list of references for recent research in the area.

Stability analysis of the zero dynamics is paramount in evaluating closed-loop stability for feedback control methods that involve plant inversion. Some plant inversion feedback control methods require stability of the zero dynamics to ensure closed-loop stability.[10] Although stability of the zero dynamics can be altered by a change in output function, redundant systems have the unique feature that their control allocation functions potentially destabilize the zero dynamics. Therefore, an output function that results in stable zero dynamics for one control allocation function could possibly result in unstable zero dynamics for another control allocation function.[11] For linear systems, concepts of invariant and controllability subspaces may be used to analyze stability of the zero dynamics with linear control allocation functions.[12] However, control allocation functions generally are nonlinear to account for actuator nonlinearities and typically are designed to optimize performance without regard to stability.[2,13,14] The main contribution of this paper is a global asymptotic stability condition for the zero dynamics with a broad class of nonlinear control allocation functions. This condition provides an analysis tool for nonlinear control allocation functions used in conjunction with feedback control methods that require stability of the zero dynamics to ensure closed-loop stability.

The paper is organized as follows. The next section introduces a two-part controller that includes control allocation and dynamic inversion of a square system. It is assumed that the control allocation has previously been designed to optimize an objective and is given. Thus, no control allocation synthesis is contained in this paper. Next, a set of admissible nonlinear control allocation functions is defined, which includes some common control allocation methods. Because stability of the zero dynamics is required for dynamic inversion to provide closed-loop stability, a global asymptotic stability condition for the zero dynamics is then derived for the set of admissible control allocation functions.

II. Controller Structure

Consider the following linear system with more controls than commanded variables:

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A_{zz} & A_{zy} \\ A_{yz} & A_{yy} \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} B_z \\ B_y \end{bmatrix} \boldsymbol{u} \tag{1}$$

$$z \in \mathbb{R}^{n_z}, \qquad \boldsymbol{y} \in \mathbb{R}^{n_y}, \qquad \boldsymbol{u} \in \mathbb{R}^{m}, \qquad n_y < m$$

where $\boldsymbol{u}$ are the system controls, $\boldsymbol{y}$ are the commanded states, and z are the uncommanded states. Assume that it is desired to design a controller that tracks $\boldsymbol{y}$ commands for a system of the form in Eq. (1). The two element controller structure of Fig. 1 is proposed. In the diagram, $\hat{k}$ is an element that addresses feedback and feedforward control of a square system derived from the system in Eq. (1), and ρ is the control allocator that addresses control redundancy. All of the control redundancy is captured in the control allocator so actuator command requirements are addressed separately from the feedback and feedforward control requirements. The separation of control allocation from the other controller elements allows the control allocator design to be performed as a subsystem design process. The benefit is that as requirements for one subsystem change, the other subsystem remains unaffected. However, control allocation

Presented as Paper 96-3691 at the AIAA Guidance, Navigation, and Control Conference, San Diego, CA, July 29–31, 1996; received March 4, 1997; revision received Nov. 12, 1997; accepted for publication Nov. 24, 1997.

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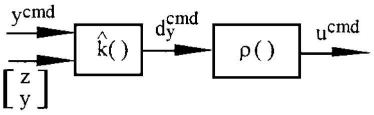

Fig. 1 Controller structure.

affects loop stability. Therefore, the disadvantage of this subsystem perspective is that control allocation effects on closed-loop stability may be neglected.

A. Control Allocation

It is useful to define the product of the control effectiveness and the controls as state derivatives due to the controls. Consider the $\boldsymbol{y}$ derivative due to the controls, which is referred to as just the $\boldsymbol{y}$ derivative $\boldsymbol{d}_y$

$$\boldsymbol{d}_y \equiv B_y \boldsymbol{u} \tag{2}$$

In the literature, many names have been used to refer to $\boldsymbol{d}_y$, which include generalized controls, fictitious controls, pseudocontrols, or reduced controls because the dimension of $\boldsymbol{d}_y$ is less than the dimension of $\boldsymbol{u}$. It becomes clear in the development of the control allocation problem why $\boldsymbol{d}_y$ is referred to as controls. Similar to the $\boldsymbol{y}$ derivative, the z derivative is defined as

$$\boldsymbol{d}_z \equiv B_z \boldsymbol{u} \tag{3}$$

Control allocation is defined by a generally nonlinear mapping from $\mathbb{R}^{n_y}$ to $\mathbb{R}^m$

$$\boldsymbol{u}^{\text{cmd}} = \rho\left(\boldsymbol{d}_y^{\text{cmd}}\right) \qquad \text{such that} \qquad B_y \rho\left(\boldsymbol{d}_y^{\text{cmd}}\right) = \boldsymbol{d}_y^{\text{cmd}} \tag{4}$$

where ρ is the control allocation function. Note that if only linear control allocation functions are considered, the control allocation problem is that of determining a right inverse of the control effectiveness matrix B_y. Thus, in general, the control allocation problem is essentially finding a nonlinear right inverse for B_y.

If $\boldsymbol{u} = \boldsymbol{u}^{\text{cmd}}$, combining the control allocation function with the redundant system of Eq. (1) results in the following system:

$$\begin{bmatrix} \dot{\boldsymbol{z}} \\ \dot{\boldsymbol{y}} \end{bmatrix} = \begin{bmatrix} A_{zz} & A_{zy} \\ A_{yz} & A_{yy} \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix} + \begin{bmatrix} B_z \rho\left(\boldsymbol{d}_y^{\text{cmd}}\right) \\ \boldsymbol{d}_y^{\text{cmd}} \end{bmatrix} \tag{5}$$

which is a good low-frequency, small control signal approximation of the open-loop behavior from $\boldsymbol{d}_y$ to $[\boldsymbol{z}\ \ \boldsymbol{y}]^T$ because typically there exist actuator dynamics and limits. If $\boldsymbol{d}_y$ is thought of as reduced controls, then this system is square. Note that the redundant system in Eq. (1) is linear yet the square system in Eq. (5) is generally nonlinear because $\rho(\boldsymbol{d}_y^{\text{cmd}})$ is possibly nonlinear. However, the square system has the nice property that the $\boldsymbol{y}$ dynamics remain linear even for nonlinear control allocation functions. The system in Eq. (5) will be used for control design, and note that the control reduction has reduced the complexity of the design because the square system in Eq. (5) has less controls than the original system.

It is initially unclear how actuator limits relate to the reduced controls of the system in Eq. (5). Actuator limits are considered by defining a limited control subset,[15] which allows extension of the actuator limits to the reduced controls,

$$\Omega \equiv \{\boldsymbol{u} \mid \underline{\boldsymbol{u}}_i \leq \boldsymbol{u}_i \leq \bar{\boldsymbol{u}}_i, i = 1, \ldots, m\} \subset \mathbb{R}^m \tag{6}$$

where $\underline{\boldsymbol{u}}$ and $\bar{\boldsymbol{u}}$ are vectors of lower and upper actuator limits, respectively. Analogously for the reduced controls, an attainable $\boldsymbol{y}$-derivative subset (ADS) is defined by the limited control subset and given by the following:

$$\Phi^y \equiv \{\boldsymbol{d}_y \mid \rho(\boldsymbol{d}_y) \in \Omega\} \tag{7}$$

where Φ^y is the ADS and ρ satisfies Eq. (4). The volume of the ADS is an indication of the $\boldsymbol{y}$-derivative capability of the system. Note that the ADS depends on the control allocation function $\rho(\boldsymbol{d}_y^{\text{cmd}})$, which means that the $\boldsymbol{y}$-derivative capability of the system will depend on the manner in which redundant controls are allocated. In other words, different control allocation functions will result in ADSs of varying volume. Algorithms exist in the literature that address the computation of the boundary of Φ^y, denoted $\partial\Phi^y$, for certain control allocation functions.[13,15] Further it is shown by Durham[15] that there exists a unique ADS of maximum volume $\Phi^y_{\max}$, which is independent of the control allocation function yet dependent on the boundary of the limited control subset $\partial\Omega$. Thus, the ADS is just the subset of reduced controls such that actuators satisfy their limits. If the control allocation function only had to satisfy Eq. (4), then any linear right inverse of B_y would suffice, and geometric approaches[12] may be used for synthesis. However, other requirements may be present such as minimal control effort, minimal computational complexity, control group prioritization, maximum $\boldsymbol{y}$-derivative capability, or some other preferred control position. All of these other requirements are referred to as allocation method requirements. These allocation method requirements typically drive the control allocation synthesis without direct regard for closed-loop stability effects. The control allocation problem is formally stated by the following.

Problem 1: control allocation problem. Find a control allocation function ρ that satisfies all allocation method requirements such that 1) $B_y \rho(\boldsymbol{d}_y^{\text{cmd}}) = \boldsymbol{d}_y^{\text{cmd}}\ \forall \boldsymbol{d}_y^{\text{cmd}} \in \Phi^y$ and 2) $B_y \rho(\boldsymbol{d}_y^{\text{cmd}}) \in \Phi^y\ \forall \boldsymbol{d}_y^{\text{cmd}} \notin \Phi^y$.

Thus, a function that solves the control allocation problem satisfies Eq. (4) for attainable derivatives ($\boldsymbol{d}_y^{\text{cmd}} \in \Phi^y$) and allocates controls that are linearly mapped by B_y to Φ^y for nonattainable derivatives ($\boldsymbol{d}_y^{\text{cmd}} \notin \Phi^y$). This makes sense because if the commanded derivative is nonattainable, then there is only hope of clipping the commanded derivative in some manner such that the control allocation output achieves the clipped derivative. Because approaches for nonattainable derivatives are highly problem dependent, the scope of this paper is limited to attainable derivatives, i.e., condition 1 of the control allocation problem.

Therefore, it is assumed for the remainder of this section that a control allocation function is given that has previously been designed to satisfy specified allocation method requirements. To complete the control law, feedforward and feedback compensation is developed in the following section.

B. Dynamic Inversion Control

Although generally nonlinear, the system in Eq. (5) provides an appropriate design model for dynamic inversion. The assumption of attainable derivatives leads to this model and allows focus on local analysis of control allocation. A controller $\hat{k}$ is now developed for the system in Eq. (5) using dynamic inversion control. Consider the following dynamic inversion control law:

$$\boldsymbol{d}_y^{\text{cmd}} = \hat{k}(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{y}^{\text{cmd}}) \equiv -A_{yz}\boldsymbol{z} - A_{yy}\boldsymbol{y} + v(\boldsymbol{y}, \boldsymbol{y}^{\text{cmd}}) \tag{8}$$

where $v(\boldsymbol{y}, \boldsymbol{y}^{\text{cmd}})$ is additional control that is possibly dynamic but is not a function of $\boldsymbol{z}$. The function $v(\boldsymbol{y}, \boldsymbol{y}^{\text{cmd}})$ is commonly referred to as the desired $\boldsymbol{y}$ dynamics and has the property $v(0, 0) = 0$. The dynamic inversion control law transforms the system in Eq. (5) into a cascade of two subsystems. The $\boldsymbol{z}$ subsystem depends on $\boldsymbol{z}$ and $\boldsymbol{y}$:

$$\dot{\boldsymbol{z}} = A_{zz}\boldsymbol{z} + A_{zy}\boldsymbol{y} + B_z\rho\left[-A_{yz}\boldsymbol{z} - A_{yy}\boldsymbol{y} + v(\boldsymbol{y}, \boldsymbol{y}^{\text{cmd}})\right] \tag{9}$$

whereas the $\boldsymbol{y}$ subsystem does not depend on $\boldsymbol{z}$:

$$\dot{\boldsymbol{y}} = v(\boldsymbol{y}, \boldsymbol{y}^{\text{cmd}}) \tag{10}$$

It is easily shown by an application of the small-gain theorem[16–18] that two cascaded asymptotically stable subsystems are asymptotically stable. The $\boldsymbol{z}$ subsystem is asymptotically stable if the system in Eq. (9) is asymptotically stable when $\boldsymbol{y}$ and $\dot{\boldsymbol{y}} = 0$. The $\boldsymbol{z}$ subsystem dynamics for $\boldsymbol{y}$ and $\dot{\boldsymbol{y}} = 0$ are called the zero dynamics. Therefore, if $v(\boldsymbol{y}, \boldsymbol{y}^{\text{cmd}})$ is chosen such that the $\boldsymbol{y}$ subsystem is asymptotically stable, then the closed-loop system is asymptotically stable if the zero dynamics are asymptotically stable. Note, however, that stability of the zero dynamics depends on the nonlinear control allocation function. The next two sections provide stability analysis of the zero dynamics, which is equivalent to closed-loop stability analysis under the standing assumption that $v(\boldsymbol{y}, \boldsymbol{y}^{\text{cmd}})$ is an asymptotically stabilizing feedback for the $\boldsymbol{y}$ subsystem.

III. Admissible Control Allocators

Vector and matrix norms are used extensively throughout the remainder of this paper. All constant real-valued vector norms will be the Euclidean norm, and all constant real-valued matrix norms will be the induced matrix two norm unless otherwise noted. The Euclidean norm for vectors and the induced matrix two norm for matrices are just the maximum singular value denoted as $\bar{\sigma}$

$$\|X\| \equiv \bar{\sigma}(X), \qquad \forall X \in \mathbb{R}^{n\times m} \tag{11}$$

In this section, admissible control allocation functions are restricted to those functions with finite z derivative L_2 gains to improve the mathematical tractability of analyzing systems with control allocators and to assist in avoiding the design of unreasonable control allocators. Recall the definition of the z derivative given by Eq. (3). If the z derivative is linearly bounded by k_{zy}, then the following inequality holds:

$$\|\boldsymbol{d}_z\| = \left\|B_z\rho\left(\boldsymbol{d}_y^{\mathrm{cmd}}\right)\right\| \le k_{zy}\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\|, \qquad \forall\, \boldsymbol{d}_y^{\mathrm{cmd}} \tag{12}$$

where k_{zy} is just the L_2 gain of $\boldsymbol{d}_z$. The L_2 gain provides the linear bound on the two norm of the z derivative. It will be shown that k_{zy} may be specified even without complete knowledge of the control allocation function.

Note that Eq. (12) implies that $\rho(0)$ belongs to the null space of B_z. However, B_z is arbitrary in the sense that there is some freedom in the choice of z in representing the input($\boldsymbol{u}$)–output($\boldsymbol{y}$) behavior of Eq. (1). For the bound in Eq. (12) to be satisfied for all possible z (and, thus, B_z) that represent the input–output behavior of the redundant system, the control allocation must have zero output for zero input, i.e., $\rho(0) = 0$.

Thus, using the z-derivative L_2 gain from Eq. (12), an admissible control allocation function is defined by the following.

Definition 1: admissible control allocation function. A control allocation function ρ is said to be admissible if it produces $\boldsymbol{d}_z$ linearly bounded by k_{zy}, which is referred to as the admissibility constant, and $\rho(0) = 0$.

Note that all admissible control allocation functions defined by an admissibility constant k_{zy} are also admissible defined by admissibility constants greater than k_{zy}. However, arbitrarily large admissibility constants provide arbitrarily conservative bounds for the z derivative in Eq. (12) which give less meaning to the stability results in Sec. IV. Three control allocation methods are now considered for development of nonconservative admissibility constants, i.e., minimal k_{zy}.

A. Control Allocation Admissibility Constants

Although every control allocation function is not guaranteed to result in linearly bounded z derivatives, it is shown that z derivatives are linearly bounded for some common control allocation methods. Admissibility constant lower bounds are developed for these control allocation functions that result in linearly bounded z derivatives.

1. Maximum ADS Control Allocation

The commanded $\boldsymbol{y}$ derivative is expressed as $\boldsymbol{d}_y^{\mathrm{cmd}} \equiv k_y\hat{\boldsymbol{d}}_y$ where $\hat{\boldsymbol{d}}_y$ is its direction and k_y is a positive scalar magnitude. The control that achieves a $\boldsymbol{y}$ derivative on the boundary of the maximum ADS in the direction of the commanded $\boldsymbol{y}$ derivative is given by

$$\boldsymbol{u}^*(\hat{\boldsymbol{d}}_y) = \left\{\boldsymbol{u} \mid B_y\boldsymbol{u} = k_y^*\hat{\boldsymbol{d}}_y\right\} \tag{13}$$

where k_y^* is the magnitude along $\hat{\boldsymbol{d}}_y$ that defines an ADS boundary point

$$k_y^* = \left\{k_y \mid k_y\hat{\boldsymbol{d}}_y \in \partial\Phi_{\max}^y\right\} \tag{14}$$

Now consider the following control allocation function $\rho_{\max}$ that achieves the maximum ADS[11,13,15]

$$\boldsymbol{u}^{\mathrm{cmd}} = \rho_{\max}\left(\boldsymbol{d}_y^{\mathrm{cmd}}\right) = \left(k_y/k_y^*\right)\boldsymbol{u}^*(\hat{\boldsymbol{d}}_y) \tag{15}$$

The form of $\rho_{\max}$ in Eq. (15) is sufficient to determine admissibility constants even though $\boldsymbol{u}^*(\hat{\boldsymbol{d}}_y)$ is an unspecified function. The following lemma states a lower bound for k_{zy} in Eq. (12) with the control allocation function $\rho_{\max}$. The lower bound is stated in terms of the columns of B_z defined as

$$B_z \equiv \begin{bmatrix}\boldsymbol{b}_{z_1} & \cdots & \boldsymbol{b}_{z_m}\end{bmatrix} \tag{16}$$

Lemma 1. Scalar k_{zy} is an admissibility constant for $\rho_{\max}$ if

$$k_{zy} \ge \frac{1}{r_{\min}}\sum_{i=1}^{m}\left\|\boldsymbol{b}_{z_i}\right\|$$

where $r_{\min}$ is defined as the minimum distance from the origin to the boundary of the maximum ADS given by

$$r_{\min} \equiv \min_{\boldsymbol{d}_y^{\mathrm{cmd}} \in \partial\Phi_{\max}^y}\left\{\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\|\right\} \tag{17}$$

Proof. Without loss of generality, assume that the controls have limits of ± 1. If this is not the case, procedures are used to normalize and symmetrize the control effectiveness matrix and the maximum ADS.[11]

Reduced controls that belong to the maximum ADS boundary are defined by

$$\boldsymbol{d}_y^* \equiv \left\{\boldsymbol{d}_y \mid \boldsymbol{d}_y \in \partial\Phi_{\max}^y\right\} \tag{18}$$

Because $r_{\min}$ is the minimum distance from the origin to $\partial\Phi_{\max}^y$, it follows that

$$r_{\min} \le \left\|\boldsymbol{d}_y^*\right\| \tag{19}$$

Multiplication of positive constants does not change the inequality

$$\frac{\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\|}{\left\|\boldsymbol{d}_y^*\right\|} \le \frac{\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\|}{r_{\min}} \tag{20}$$

The normal and symmetric control limits imply that $|\boldsymbol{u}_i^*| \le 1$. Therefore, the following holds:

$$\frac{\left|\boldsymbol{u}_i^*\right|}{\left\|\boldsymbol{d}_y^*\right\|}\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\| \le \frac{\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\|}{r_{\min}} \tag{21}$$

Again using the fact that multiplication by positive quantities does not change inequalities, it follows that

$$\left\|\boldsymbol{b}_{z_i}\right\|\frac{\left|\boldsymbol{u}_i^*\right|}{\left\|\boldsymbol{d}_y^*\right\|}\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\| \le \frac{\left\|\boldsymbol{b}_{z_i}\right\|}{r_{\min}}\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\| \tag{22}$$

Note that summation over all i does not change the inequality, so that the following holds:

$$\sum_{i=1}^{m}\left\|\boldsymbol{b}_{z_i}\right\|\frac{\left|\boldsymbol{u}_i^*\right|}{\left\|\boldsymbol{d}_y^*\right\|}\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\| \le \sum_{i=1}^{m}\frac{\left\|\boldsymbol{b}_{z_i}\right\|}{r_{\min}}\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\| \tag{23}$$

The lemma statement implies the following:

$$\sum_{i=1}^{m}\left\|\boldsymbol{b}_{z_i}\right\|\frac{\left|\boldsymbol{u}_i^*\right|}{\left\|\boldsymbol{d}_y^*\right\|}\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\| \le k_{zy}\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\| \tag{24}$$

From the definition of $\rho_{\max}$ in Eq. (15), the left-hand side becomes

$$\sum_{i=1}^{m}\left\|\boldsymbol{b}_{z_i}\right\|\left|\rho_{\max_i}\left(\boldsymbol{d}_y^{\mathrm{cmd}}\right)\right| = \sum_{i=1}^{m}\left\|\boldsymbol{b}_{z_i}\right\|\frac{\left|\boldsymbol{u}_i^*\right|}{\left\|\boldsymbol{d}_y^*\right\|}\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\| \le k_{zy}\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\| \tag{25}$$

Because

$$\left\|B_z\rho_{\max}\left(\boldsymbol{d}_y^{\mathrm{cmd}}\right)\right\| \le \sum_{i=1}^{m}\left\|\boldsymbol{b}_{z_i}\right\|\left|\rho_{\max_i}\left(\boldsymbol{d}_y^{\mathrm{cmd}}\right)\right| \tag{26}$$

it is concluded that

$$\|\boldsymbol{d}_z\| = \left\|B_z\rho_{\max}\left(\boldsymbol{d}_y^{\mathrm{cmd}}\right)\right\| \le k_{zy}\left\|\boldsymbol{d}_y^{\mathrm{cmd}}\right\| \tag{27}$$

which implies that k_{zy} is an admissibility constant for $\rho_{\max}$.

2. Linear Control Allocation

Linear control allocation functions $\rho_{\rm lin}$ take the following form[11]:

$$u^{\rm cmd} = \rho_{\rm lin}\left(d_y^{\rm cmd}\right) = R\, d_y^{\rm cmd}, \qquad B_y R = I_{n_y} \tag{28}$$

where I_{n_y} is the $n_y \times n_y$ identity. For linear control allocation functions, the stability of the zero dynamics is completely characterized by the eigenvalues of the matrix $A_{zz} - B_z R A_{yz}$ [see Eq. (9)]. However, it may be useful to analyze zero dynamics stability for an entire class of admissible control allocation functions, linear and nonlinear. For such situations, a lower bound for k_{zy} in Eq. (12) with the control allocation function $\rho_{\rm lin}$ is stated in the following lemma.

Lemma 2. Scalar k_{zy} is an admissibility constant for $\rho_{\rm lin}$ if $k_{zy} \geq \|B_z R\|$.

Proof. The following inequality is implied by the lemma statement:

$$\|B_z R\| \left\|d_y^{\rm cmd}\right\| \leq k_{zy} \left\|d_y^{\rm cmd}\right\| \tag{29}$$

Using the properties of matrix and vector norms implies the following expression which is just Eq. (12) with the control allocation $\rho_{\rm lin}$:

$$\|d_z\| = \left\|B_z R d_y^{\rm cmd}\right\| \leq k_{zy} \left\|d_y^{\rm cmd}\right\| \tag{30}$$

Therefore k_{zy} is an admissibility constant for $\rho_{\rm lin}$.

3. Daisy-Chain Control Allocation

This section develops lower bounds for the admissibility constant corresponding to the daisy-chain control allocation method. The daisy chain commands redundant groups of limited controls in a prioritized manner. The first control group is linearly allocated, and subsequent groups are zero until a limit is encountered by the first control group. When the first control group encounters limits, the second control group is allocated linearly to make up the command difference resulting from limits on the first control group. Daisy-chain control allocation is given by the following when there are two redundant groups of controls[11]:

$$u^{\rm cmd} = \rho_{dc}\left(d_y^{\rm cmd}\right) = \begin{bmatrix} SAT_1\left(R_1 d_y^{\rm cmd}\right) \\ R_2\left\{d_y^{\rm cmd} - B_{y_1} SAT_1\left(R_1 d_y^{\rm cmd}\right)\right\} \end{bmatrix} = \begin{bmatrix} \mu_1^{\rm cmd} \\ \mu_2^{\rm cmd} \end{bmatrix} \tag{31}$$

where SAT_1 is an $m_1 \times 1$ vector of standard saturation functions shown in Fig. 2, m_1 is the number of elements in the first control group, and R_i is any right inverse of B_{y_i}, i.e., $B_{y_i} R_i = I_{n_y}$ for $i = 1, 2$. The following lemma states admissibility constant lower bounds for ρ_{dc}.

Lemma 3. Scalar k_{zy} is an admissibility constant for ρ_{dc} if

$$k_{zy} \geq \left\|B_{z_2} R_2\right\| + \left\|B_{z_1} - B_{z_2} R_2 B_{y_1}\right\| \|R_1\|$$

Proof. Start with the definition of the z derivative with the daisy-chain allocator

$$\begin{aligned} d_z &= B_z \rho_{dc}\left(d_y^{\rm cmd}\right) \\ &= B_{z_1} SAT_1\left(R_1 d_y^{\rm cmd}\right) + B_{z_2} R_2\left\{d_y^{\rm cmd} - B_{y_1} SAT_1\left(R_1 d_y^{\rm cmd}\right)\right\} \end{aligned} \tag{32}$$

Utilizing the properties of vector and matrix norms gives the following bound:

$$\|d_z\| \leq \left\|B_{z_2} R_2\right\| \left\|d_y^{\rm cmd}\right\| + \left\|B_{z_1} - B_{z_2} R_2 B_{y_1}\right\| \left\|SAT_1\left(R_1 d_y^{\rm cmd}\right)\right\| \tag{33}$$

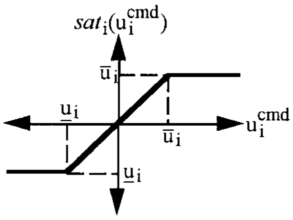

Fig. 2 Standard saturation function.

Noting that $\|SAT_1(R_1 d_y^{\rm cmd})\| \leq \|R_1\| \|d_y^{\rm cmd}\|$, the following bound is derived by collecting terms:

$$\|d_z\| \leq \left\{\left\|B_{z_2} R_2\right\| + \left\|B_{z_1} - B_{z_2} R_2 B_{y_1}\right\| \|R_1\|\right\} \left\|d_y^{\rm cmd}\right\| \tag{34}$$

The lemma statement gives the following bound:

$$\|d_z\| \leq k_{zy} \left\|d_y^{\rm cmd}\right\| \tag{35}$$

which implies that k_{zy} is an admissibility constant for ρ_{dc}.

B. Discussion and Examples

Suppose it is desired to define a set of control allocation functions by a single admissibility constant that includes the maximum ADS, linear, and daisy-chain control allocators. The least conservative admissible set is defined by the following admissibility constant:

$$k_{zy} = \max\left\{\frac{1}{r_{\min}} \sum_{i=1}^{m} \left\|b_{z_i}\right\|, \|B_z R\|, \left\|B_{z_2} R_2\right\| + \left\|B_{z_1} - B_{z_2} R_2 B_{y_1}\right\| \|R_1\|\right\} \tag{36}$$

which is the maximum of the lower bounds corresponding to each individual method.

The following examples demonstrate the maximum ADS, linear, and daisy-chain control allocator admissibility lemmas. For linear allocators, the following examples consider only the special case of identity weighted pseudoinverse control allocators[11]:

$$u^{\rm cmd} = R_{\rm pseu} d_y^{\rm cmd} = B_y^T\left(B_y B_y^T\right)^{-1} d_y^{\rm cmd} \tag{37}$$

An aircraft longitudinal axis flight control example is considered first, and the second example considers aircraft lateral/directional axes flight control. In both examples it is desired to find a set of control allocation functions, defined by the least conservative admissibility constant, that includes the maximum ADS, pseudoinverse, and daisy-chain methods. This set will assist in the zero dynamics stability analysis developed in Sec. IV.

Example 1: aircraft longitudinal axis admissible allocators. Consider the short-period approximation of the longitudinal dynamics of an F-18 supermaneuverable aircraft at Mach 0.6, 30,000-ft altitude, and 5.2-deg angle of attack[3]

$$\begin{bmatrix} \dot\alpha \\ \dot q \end{bmatrix} = \begin{bmatrix} -0.509 & 0.994 \\ -1.13 & -0.280 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} -0.093 & -0.018 \\ -6.57 & -1.53 \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_{\rm ptv} \end{bmatrix} \tag{38}$$

The states are angle of attack α in degrees and body-axis pitch rate q in degrees per second. The controls are elevator δ_e in degrees and pitch thrust vector angle $\delta_{\rm ptv}$ in degrees. The controls have the following limits:

$$\begin{bmatrix} -23.07 \\ -30 \end{bmatrix} \leq \begin{bmatrix} \delta_e \\ \delta_{\rm ptv} \end{bmatrix} \leq \begin{bmatrix} 11.43 \\ 30 \end{bmatrix} \tag{39}$$

The pilot command is chosen to be q, and the state partition is taken to be

$$z = \alpha, \qquad y = q \tag{40}$$

in terms of the system in Eq. (1). Assume that it is desired to use the thrust vectoring nozzle only in the event of elevator saturation, which is interpreted as an allocation method requirement. The daisy-chain control allocation commands the elevator and thrust vectoring such that the allocation method requirement restricting the use of thrust vectoring is satisfied. Because there are no zero elements in the control effectiveness matrix and $n_y = 1$, the following control groups are redundant:

$$\mu_1 = \delta_e, \qquad \mu_2 = \delta_{\rm ptv} \tag{41}$$

The state and control partitions lead to the following control effectiveness partitions:

$$B_y = \begin{bmatrix} b_{y_1} & b_{y_2} \end{bmatrix} = [-6.57 \quad -1.53]$$
$$B_z = \begin{bmatrix} b_{z_1} & b_{z_2} \end{bmatrix} = [-0.093 \quad -0.018] \tag{42}$$

The control effectiveness matrices are normalized such that the control limits are ± 1

$$B_{y_N} = \begin{bmatrix} b_{y_{N_1}} & b_{y_{N_2}} \end{bmatrix} = [-113.3 \quad -45.8]$$
$$B_{z_N} = \begin{bmatrix} b_{z_{N_1}} & b_{z_{N_2}} \end{bmatrix} = [-1.60 \quad -0.540] \tag{43}$$

Note this normalization is required for application of lemma 1. Because the number of y states is unity ($n_y = 1$) in this example, computation of the symmetric maximum ADS is trivial:

$$\Phi^y_{\max_s} = \{d_{y_s} | |d_{y_s}| \leq 159.2\} \tag{44}$$

Therefore, the minimum radius of $\partial \Phi^y_{\max_s}$ is

$$r_{\min} = 159.2 \tag{45}$$

The admissibility constant lower bound for $\rho_{\max}$ is given by

$$\frac{1}{r_{\min}} \sum_{i=1}^{m} \left| b_{z_{N_i}} \right| = 0.013 \tag{46}$$

from lemma 1. Thus, if k_{zy} is chosen greater than or equal to 0.013, then it is an admissibility constant for $\rho_{\max}$ in Eq. (15). Now consider pseudoinverse control allocation. Because the pseudoinverse control allocation is linear, the lower bound from lemma 2 is used

$$\|B_z R_{\text{pseu}}\| = \left\| B_z B_y^T \left(B_y B_y^T\right)^{-1} \right\| = 0.014 \tag{47}$$

Thus, if k_{zy} is chosen greater than or equal to 0.014, then it is an admissibility constant for $R_{\text{pseu}} d_y$. Now consider the daisy-chain control allocation with assistance from lemma 3. The lower bound in lemma 3 is given by the following:

$$\left| \frac{b_{z_2}}{b_{y_2}} \right| + \left| b_{z_1} - \frac{b_{z_2} b_{y_1}}{b_{y_2}} \right| \left| \frac{1}{b_{y_1}} \right| = 0.014 \tag{48}$$

Thus, if k_{zy} is chosen greater than or equal to 0.014, then it is an admissibility constant for ρ_{dc} in Eq. (31). Therefore, the maximum ADS, pseudoinverse, and daisy-chain control allocation functions all belong to an admissible set defined by $k_{zy} \geq 0.014$, and the least conservative admissible set is defined by $k_{zy} = 0.014$.

Example 2: aircraft lateral/directional axes admissible allocators. Now consider the following linear approximation of the lateral/directional dynamics of an F-18 supermaneuverable aircraft at Mach 0.6, 30,000-ft altitude, and 5.2-deg angle of attack[3]

$$\begin{bmatrix} \dot\beta \\ \dot p \\ \dot r \end{bmatrix} = \begin{bmatrix} -0.112 & 0.094 & -0.995 \\ -10.2 & -1.17 & 0.432 \\ 2.20 & -0.010 & -0.106 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \end{bmatrix} + \begin{bmatrix} -0.009 & -0.005 & 0.013 & 0 & 0.008 \\ 9.02 & 11.06 & 0.842 & 0.730 & 0.055 \\ 0.296 & -0.264 & -0.684 & -0.002 & -0.757 \end{bmatrix} \begin{bmatrix} \delta_{\text{dt}} \\ \delta_a \\ \delta_r \\ \delta_{\text{rtv}} \\ \delta_{\text{ytv}} \end{bmatrix} \tag{49}$$

The states are sideslip angle β, body-axis roll rate p, and body-axis yaw rate r, with the angles expressed in degrees and angular rates in degrees per second. The controls are differential horizontal tail δ_{dt}, aileron δ_a, rudder δ_r, roll thrust vector angle δ_{rtv}, and yaw thrust vector angle δ_{ytv}, which are all expressed in degrees. The control effectors have a limited range of operation defined by the following:

$$\begin{bmatrix} -17.25 \\ -35 \\ -30 \\ -30 \\ -30 \end{bmatrix} \leq \begin{bmatrix} \delta_{\text{dt}} \\ \delta_a \\ \delta_r \\ \delta_{\text{rtv}} \\ \delta_{\text{ytv}} \end{bmatrix} \leq \begin{bmatrix} 17.25 \\ 35 \\ 30 \\ 30 \\ 30 \end{bmatrix} \tag{50}$$

It is desired to command the roll and yaw rates, which leads to the following state partition in terms of the general system in Eq. (1)

$$z = \beta, \qquad \boldsymbol{y} = [p \quad r]^T \tag{51}$$

As in example 1, assume that it is desired to use the thrust vectoring nozzles only in the event of aerodynamic control surface saturation, which again is interpreted as an allocation method requirement. This naturally leads to the following control group partition:

$$\begin{bmatrix} \boldsymbol{\mu}_1 \\ \hline \boldsymbol{\mu}_2 \end{bmatrix} = \begin{bmatrix} \delta_{\text{dt}} \\ \delta_a \\ \delta_r \\ \hline \delta_{\text{rtv}} \\ \delta_{\text{ytv}} \end{bmatrix} \tag{52}$$

The state and control partitions lead to the following control effectiveness matrices:

$$B_y = \begin{bmatrix} 9.02 & 11.06 & 0.842 & 0.730 & 0.055 \\ 0.296 & -0.264 & -0.684 & -0.002 & -0.757 \end{bmatrix}$$
$$B_z = [-0.009 \quad -0.005 \quad 0.013 \quad 0 \quad 0.008] \tag{53}$$

The normalized control effectiveness matrices are given by the following:

$$B_{y_N} = \begin{bmatrix} 156.0 & 387.0 & 25.3 & 21.9 & 1.66 \\ 5.10 & -9.23 & -20.5 & -0.058 & -22.7 \end{bmatrix}$$
$$B_{z_N} = [-0.159 \quad -0.171 \quad 0.379 \quad 0 \quad 0.250] \tag{54}$$

Note again that normalization is required for lemma 1. Because the control limits are symmetric about the origin, no symmetrization is necessary for $\partial\Phi^y_{\max}$. Computation of the boundary $\partial\Phi^y_{\max}$ is a problem of finding the convex hull for a set of points. This problem is addressed by Durham[13,15,19] and is beyond the scope of this document. An approximation to $\partial\Phi^y_{\max}$ is found by parameterizing the two-dimensional $\boldsymbol{y}$-derivative space with polar coordinates, i.e., a magnitude and an angle. This parameterization is used in this example and the approximation of $\partial\Phi^y_{\max}$ is shown in Fig. 3. The minimum radius of $\partial\Phi^y_{\max}$ is determined graphically and used to define a ball in $\boldsymbol{d}_y$ space

$$r_{\min} = 52.0, \qquad B_{r_{\min}} \equiv \{\boldsymbol{d}_y | \|\boldsymbol{d}_y\| \leq r_{\min}\} \tag{55}$$

The boundary of the ball $\partial B_{r_{\min}}$ is also shown in Fig. 3 and appears as an ellipse because the axes have different scales. The admissibility constant lower bound for $\rho_{\max}$ is given by the following:

$$\frac{1}{r_{\min}} \sum_{i=1}^{m} \left| b_{z_{N_i}} \right| = 0.018 \tag{56}$$

from lemma 1. Thus, if k_{zy} is chosen greater than or equal to 0.018, then it is an admissibility constant for $\rho_{\max}$. Now consider pseudoinverse control allocation. The lower bound from lemma 2 is used for the pseudoinverse allocator because it is linear:

$$\|B_z R_{\text{pseu}}\| = \left\| B_z B_y^T \left(B_y B_y^T\right)^{-1} \right\| = 0.014 \tag{57}$$

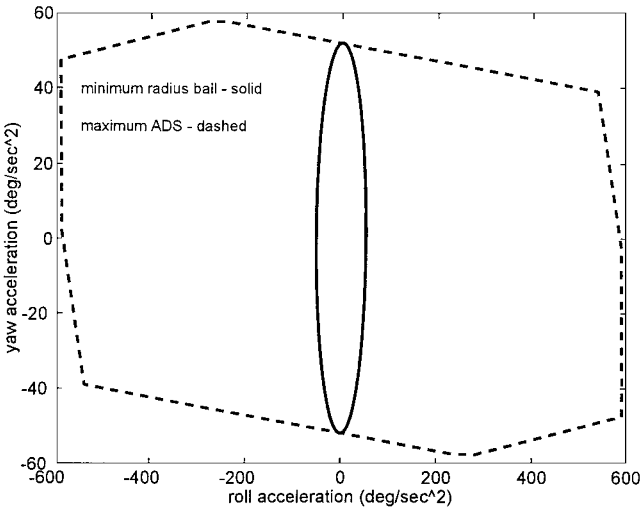

Fig. 3 Maximum ADS.

Thus, if k_{zy} is chosen greater than or equal to 0.014, then it is an admissibility constant for $R_{\mathrm{pseu}}\boldsymbol{d}_y$. Now consider daisy-chain allocation. The lower bound from lemma 3 is used to define an admissibility constant

$$\left\|B_{z_2}R_2\right\| + \left\|B_{z_1} - B_{z_2}R_2B_{y_1}\right\|\|R_1\| = 0.025 \tag{58}$$

Thus, if k_{zy} is chosen greater than or equal to 0.025, then it is an admissibility constant for ρ_{dc}. Therefore, the maximum ADS, pseudoinverse, and daisy-chain control allocation functions all belong to an admissible set defined by $k_{zy} \geq 0.025$, and the least conservative admissibility constant is $k_{zy} = 0.025$.

It has been shown how to construct the least conservative admissibility constant that defines a set of control allocation functions that includes the maximum ADS, linear, and daisy-chain control allocators. Now a sufficient condition for globally asymptotically stable zero dynamics is derived for the admissible set of control allocation functions to assist in analyzing the closed-loop stability of systems controlled by plant inversion techniques such as dynamic inversion or feedback linearization.

IV. Globally Asymptotically Stable Zero Dynamics

Because global asymptotic stability of the zero dynamics is required for some plant inversion control techniques such as dynamic inversion to achieve closed-loop stability, a sufficient condition for global asymptotic stability of the zero dynamics is derived in this section. The result is derived from direct application of the small-gain theorem,[16–18] which states that feedback loops consisting of stable operators are stable if the product of the operator gains are strictly less than unity.

The following result utilizes the small-gain theorem to ensure global asymptotic stability of the zero dynamics of the system in Eq. (5) with respect to input $\boldsymbol{d}_y$ and output $\boldsymbol{y}$. Recall that the zero dynamics are the system dynamics along the zero output manifold, i.e., the dynamics when the output $\boldsymbol{y}$ is constrained to be zero. The input required to maintain zero output is found by examining the $\boldsymbol{y}$ dynamics of Eq. (5)

$$\begin{gathered}\dot{\boldsymbol{y}} = A_{yz}\boldsymbol{z} + A_{yy}\boldsymbol{y} + \boldsymbol{d}_y \\ \boldsymbol{y} = 0, \qquad \dot{\boldsymbol{y}} = 0 \Rightarrow \boldsymbol{d}_y = -A_{yz}\boldsymbol{z}\end{gathered} \tag{59}$$

Inserting $\boldsymbol{y} = 0$ and $\boldsymbol{d}_y$ from Eq. (59) into the $\boldsymbol{z}$ dynamics of Eq. (5) gives the zero dynamics

$$\dot{\boldsymbol{z}} = A_{zz}\boldsymbol{z} + B_z\rho(-A_{yz}\boldsymbol{z}) \tag{60}$$

Therefore, the zero dynamics with respect to input $\boldsymbol{d}_y$ and output $\boldsymbol{y}$ of the system in Eq. (5) depend on the control allocation function ρ. It can be shown[9] that the control allocation can introduce unstable zero dynamics into the square system of Eq. (5) and, therefore, destabilize a closed-loop system that is controlled by dynamic inversion.

Consider the zero dynamics in Eq. (60) rewritten as an interconnection of two subsystems.

System 1:

$$\dot{\boldsymbol{z}} = A_{zz}\boldsymbol{z} + \boldsymbol{w}_2, \qquad \boldsymbol{w}_1 = -A_{yz}\boldsymbol{z} \tag{61a}$$

System 2:

$$\boldsymbol{w}_2 = B_z\rho(\boldsymbol{w}_1) \tag{61b}$$

Assuming that A_{zz} is Hurwitz, the L_2 gain of system 1 is its H_∞ norm

$$\|\boldsymbol{w}_1\| = \left\|A_{yz}(sI - A_{zz})^{-1}\right\|_\infty \|\boldsymbol{w}_2\| \tag{62}$$

and the L_2 gain of system 2 is just the admissibility constant k_{zy}

$$\|\boldsymbol{w}_2\| = k_{zy}\|\boldsymbol{w}_1\| \tag{63}$$

Therefore, asymptotic stability of the zero dynamics is guaranteed if the following small-gain condition is satisfied:

$$\left\|A_{yz}(sI - A_{zz})^{-1}\right\|_\infty k_{zy} < 1 \tag{64}$$

In summary, the small gain condition in Eq. (64) is sufficient to ensure global asymptotic stability of the zero dynamics for systems with admissible control allocation functions defined by some given admissibility constant. The following examples demonstrate the utility of the result. Assume for all of the examples that an original system of the form given in Eq. (1) is to be stabilized by plant inversion control techniques, such as dynamic inversion, which require asymptotic stability of the zero dynamics to guarantee closed-loop stability.

Example 3: aircraft longitudinal axis zero dynamics. This example is a continuation of example 1. In example 1, the choice of z and y result in the following data:

$$A_{zz} = -0.509, \qquad A_{yz} = -1.13 \tag{65}$$

Example 1 also suggests that an admissible set of functions be defined by $k_{zy} = 0.014$ to include the maximum ADS, pseudoinverse, and daisy-chain control allocation methods in the admissible set. The admissibility constant $k_{zy} = 0.014$ satisfies the small-gain condition in Eq. (64)

$$\left\|A_{yz}(sI - A_{zz})^{-1}\right\|_\infty k_{zy} = (2.22)(0.014) = 0.031 < 1 \tag{66}$$

It is concluded that the zero dynamics are globally asymptotically stable for the maximum ADS, pseudoinverse, and daisy-chain control allocation functions. In fact, Eq. (66) also shows that the zero dynamics are globally asymptotically stable for all control allocation functions belonging to the admissible set defined by $k_{zy} < 1/2.22 = 0.450$. Therefore, the combination of a dynamic inversion controller for the following square system:

$$\begin{bmatrix}\dot{\alpha} \\ \dot{q}\end{bmatrix} = \begin{bmatrix}-0.509 & 0.994 \\ -1.13 & -0.280\end{bmatrix}\begin{bmatrix}\alpha \\ q\end{bmatrix} + \begin{bmatrix}0 \\ d_q\end{bmatrix} \tag{67}$$

and an admissible control allocation function defined by $k_{zy} < 0.450$ will globally stabilize the redundant system in Eq. (38).

Example 4: aircraft lateral/directional axes zero dynamics. This example is a continuation of example 2. In example 2, the choice of z and $\boldsymbol{y}$ result in the following data:

$$A_{zz} = -0.112, \qquad A_{yz} = \begin{bmatrix}-10.2 \\ 2.20\end{bmatrix} \tag{68}$$

Example 2 also suggests that an admissible set of functions be defined by $k_{zy} = 0.025$ to include the maximum ADS, pseudoinverse, and daisy-chain control allocation methods. The admissibility constant $k_{zy} = 0.025$ does not satisfy the small-gain condition in Eq. (64)

$$\left\|A_{yz}(sI - A_{zz})^{-1}\right\|_\infty k_{zy} = (93.2)(0.025) = 2.33 > 1 \tag{69}$$

It is concluded that the zero dynamics may not be globally asymptotically stable for all admissible control allocation functions defined by $k_{zy} = 0.025$. In fact, the small-gain condition is not satisfied for any of the admissibility constants corresponding to each allocation method because

$$\left\| A_{yz}(sI - A_{zz})^{-1} \right\|_\infty = 93.2 > \frac{1}{0.014} > \frac{1}{0.018} > \frac{1}{0.025} \tag{70}$$

It is concluded that the maximum ADS, pseudoinverse, and daisy-chain control allocators may destabilize the zero dynamics. The zero dynamics are guaranteed to be globally asymptotically stable only for control allocation functions belonging to an admissible set defined by

$$k_{zy} < 1/93.2 = 0.011 \tag{71}$$

Because the sufficient condition for stability is not satisfied for example 2, more analysis must be done for the specific control allocation function that will be used with the dynamic inversion controller. Additional analysis for a particular control allocation method may be difficult, e.g., highly nonlinear control allocation methods, or simple, e.g., linear control allocation methods, depending on the method. The next example demonstrates sufficiency of the theorem.

Example 5: sufficiency of zero dynamics stability theorem. Consider the redundant system given by the following:

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -0.06 & A_{zy} \\ 0.51 & A_{yy} \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} -0.4 & 0.3 \\ 1.34 & 1.13 \end{bmatrix} \boldsymbol{u} \tag{72}$$

It is desired to know if the following linear control allocator will destabilize the zero dynamics:

$$\begin{aligned} \rho(d_y) &= Rd_y, \qquad R = [0.100 \quad 0.766]^T \\ B_y R &= [1.34 \quad 1.13][0.100 \quad 0.766]^T = 1 \end{aligned} \tag{73}$$

The lower bound from lemma 2 is used to construct an admissibility constant that defines an admissible set of functions that includes the linear allocation function

$$\|B_z R\| = \left| [-0.4 \quad 0.3] \begin{bmatrix} 0.100 \\ 0.766 \end{bmatrix} \right| = 0.190 \tag{74}$$

Therefore, if an admissible set of control allocation functions is defined by $k_{zy} = 0.190$, then Rd_y belongs to the admissible set. However, the small-gain condition is not satisfied:

$$\left\| A_{yz}(sI - A_{zz})^{-1} \right\|_\infty k_{zy} = \left\| \frac{0.51}{s + 0.06} \right\|_\infty (0.190) = 1.62 > 1 \tag{75}$$

As in example 4, it is concluded that the zero dynamics may not be globally asymptotically stable for all admissible control allocation functions defined by $k_{zy} = 0.190$. Because the control allocation is linear, the zero dynamics are linear. The stability of the zero dynamics is, thus, determined by the sign of the following scalar:

$$A_{zz} - B_z R A_{yz} = -0.06 - [-0.4 \quad 0.3] \begin{bmatrix} 0.100 \\ 0.766 \end{bmatrix} (0.51) = -0.157 \tag{76}$$

It is concluded that the zero dynamics are globally asymptotically stable because $A_{zz} - B_z R A_{yz} < 0$ even though the small-gain condition is not satisfied. Therefore, it is seen that stability analysis of the zero dynamics provided by the small-gain condition may be conservative. This example reminds the user that the theorem only provides sufficient conditions. Note that for this linear example, the additional analysis of checking the sign of $A_{zz} - B_z R A_{yz}$ would probably have been the primary zero dynamics stability analysis if only Rd_y is considered for control allocation.

V. Conclusions

A class of admissible nonlinear control allocation functions that encompasses many common control allocation methods is defined. A sufficient condition for globally asymptotically stable zero dynamics is derived for redundant systems with admissible control allocation functions. The stability condition may be conservative due to its reliance on the small-gain theorem. However, aircraft examples demonstrate the utility of the result to analyze control allocation and closed-loop stability interactions.

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